

CHARACTERIZATION AND THE PRE-SCHWARZIAN NORM ESTIMATE FOR CONCAVE UNIVALENT FUNCTIONS

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ABSTRACT. Let $Co(\alpha)$ denote the class of concave univalent functions in the unit disk \mathbb{D} . Each function $f \in Co(\alpha)$ maps the unit disk \mathbb{D} onto the complement of an unbounded convex set. In this paper we find the exact disk of variability for the functional $(1 - |z|^2)(f''(z)/f'(z))$, $f \in Co(\alpha)$. In particular, this gives sharp upper and lower estimates for the pre-Schwarzian norm of concave univalent functions. Next we obtain the set of variability of the functional $(1 - |z|^2)(f''(z)/f'(z))$, $f \in Co(\alpha)$ whenever $f''(0)$ is fixed. We also give a characterization for concave functions in terms of Hadamard convolution. In addition to sharp coefficient inequalities, we prove that functions in $Co(\alpha)$ belong to the H^p space for $p < 1/\alpha$.

1. INTRODUCTION AND PRELIMINARY RESULTS

Let \mathcal{H} denote the class of functions analytic in the unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$. We denote the class of locally univalent functions by \mathcal{LU} . The class of locally univalent functions is a vector space with respect to Hornich operations (see [8]). For $f \in \mathcal{LU}$, the pre-Schwarzian derivative T_f is defined by

$$T_f = \frac{f''}{f'}$$

and we define the norm of T_f by

$$\|T_f\| = \sup_{z \in \mathbb{D}} (1 - |z|^2) |T_f(z)|.$$

This is indeed a norm with respect to Hornich operations. It is known that $\|f\| < \infty$ if and only if f is uniformly locally univalent, i.e. there exists a constant $r = r(f) > 0$ such that f is univalent in each disk of hyperbolic radius r in \mathbb{D} .

Let \mathcal{A} denote the class of functions $f \in \mathcal{H}$ with the normalization $f(0) = f'(0) - 1 = 0$ and \mathcal{S} be the class of functions in \mathcal{A} that are univalent in \mathbb{D} and \mathcal{H}_1 denotes the class of functions f in \mathcal{H} such that $f(0) = 1$. Also we define the subclass $\mathcal{K} \subset \mathcal{S}$ of convex functions whenever $f(\mathbb{D})$ is a convex domain and the subclass \mathcal{S}^* of starlike functions whenever $f(\mathbb{D})$ is a domain that is starlike with respect to the origin (cf. [7, 11]). It is well known that $\|T_f\| \leq 6$ for $f \in \mathcal{S}$, and $\|T_f\| \leq 4$ for $f \in \mathcal{K}$. Conversely, by Becker's theorem ([3]) it follows that if $f \in \mathcal{A}$ and $\|T_f\| \leq 1$ then $f \in \mathcal{S}$.

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A function $f : \mathbb{D} \rightarrow \mathbb{C}$ is said to belong to the family $Co(\alpha)$ if f satisfies the following conditions:

- (i) f is analytic in \mathbb{D} with the standard normalization $f(0) = f'(0) - 1 = 0$. In addition it satisfies $f(1) = \infty$.
- (ii) f maps \mathbb{D} conformally onto a set whose complement with respect to \mathbb{C} is convex.
- (iii) the opening angle of $f(\mathbb{D})$ at ∞ is less than or equal to $\pi\alpha$, $\alpha \in (1, 2]$.

This paper concerns the family $Co(\alpha)$ and in order to proceed with our investigation, we recall the analytic characterization for functions in $Co(\alpha)$, $\alpha \in (1, 2]$: $f \in Co(\alpha)$ if and only if

$$(1.1) \quad \operatorname{Re} P_f(z) > 0, \quad z \in \mathbb{D},$$

where

$$P_f(z) = \frac{2}{\alpha - 1} \left[\frac{(\alpha + 1)}{2} \frac{1 + z}{1 - z} - 1 - z \frac{f''(z)}{f'(z)} \right].$$

The class $Co(\alpha)$ is referred to as the class of concave univalent functions and for a detailed discussion about concave functions, we refer to [1, 2, 6]. We note that for $f \in Co(\alpha)$, $\alpha \in (1, 2]$, the closed set $\mathbb{C} \setminus f(\mathbb{D})$ is convex and unbounded. We observe that $Co(2)$ contains the classes $Co(\alpha)$, $\alpha \in (1, 2]$.

In this paper, we first find the exact set of variability for the functional $(1 - |z|^2)T_f(z)$ and as a consequence of this we derive upper and lower bounds for the pre-Schwarzian norm $\|T_f\|$, for functions f in $Co(\alpha)$. Next we obtain the set of variability of the functional $(1 - |z|^2)T_f(z)$, $f \in Co(\alpha)$ whenever $f''(0)$ is fixed. Also, we give a representation formula in terms of Hadamard convolution for functions in $Co(\alpha)$ and some interesting link with the Kaplan class. Lastly, we present sharp inequalities among coefficients of functions in $Co(\alpha)$.

2. MAIN RESULTS

First we prove the following lemma:

Lemma 2.1. *Let $\psi \in \mathcal{H}_1$ be such that it is starlike with respect to 1 and suppose that $g \in \mathcal{A}$ satisfies*

$$\frac{2}{\alpha - 1} \left[\frac{(\alpha + 1)}{2} \frac{1 + z}{1 - z} - 1 - z \frac{g''(z)}{g'(z)} \right] = \psi(z), \quad z \in \mathbb{D},$$

for some $\alpha \in (1, 2]$. Then, for $f \in Co(\alpha)$, the condition

$$(2.2) \quad \frac{2}{\alpha - 1} \left[\frac{(\alpha + 1)}{2} \frac{1 + z}{1 - z} - 1 - z \frac{f''(z)}{f'(z)} \right] \prec \psi(z)$$

implies $(1 - z)^{\alpha+1} f'(z) \prec (1 - z)^{\alpha+1} g'(z)$.

Proof. We first note that $\psi(0) = 1$ and

$$\psi'(0) = \frac{2}{\alpha - 1} [(\alpha + 1) - g''(0)],$$

as ψ is starlike and hence univalent. Also, we note that

$$g'(z) = \exp \int_0^z \frac{(\alpha - 1)[1 - (1 - \zeta)\psi(\zeta)] + (\alpha + 3)\zeta}{2\zeta(1 - \zeta)} d\zeta, \quad z \in \mathbb{D},$$

which is a non-vanishing analytic function in the unit disk. Let

$$\begin{aligned} h(z) &= \frac{2}{(\alpha - 1)\psi'(0)} [-\log((1 - z)^{\alpha+1}g'(z))] \\ &:= -c \log((1 - z)^{\alpha+1}g'(z)), \quad c = \frac{2}{(\alpha - 1)\psi'(0)}. \end{aligned}$$

Since $\frac{(\alpha-1)c}{2}(\psi - 1) \in \mathcal{A}$ is starlike, a computation shows that

$$1 + z \frac{h''(z)}{h'(z)} = \frac{z\psi'(z)}{\psi(z) - 1}$$

has positive real part and so $h(z)$ is convex with $h(0) = 0 = h'(0) - 1$. The condition (2.2) and a little computation reveals that

$$c \left[(\alpha + 1) \frac{z}{1 - z} - z \frac{f''(z)}{f'(z)} \right] \prec c \left[(\alpha + 1) \frac{z}{1 - z} - z \frac{g''(z)}{g'(z)} \right] = zh'(z).$$

Equivalently the above can be written as

$$z [-c \log((1 - z)^{\alpha+1}f'(z))] \prec zh'(z).$$

As $h(z)$ is convex, by using a result due to Suffridge [10, p. 76, Theorem 3.1d], we get

$$-c \log((1 - z)^{\alpha+1}f'(z)) \prec h(z) = -c \log((1 - z)^{\alpha+1}g'(z)),$$

which gives the desired result. \square

We now recall that, for $f, g \in \mathcal{A}$, the condition $f' \prec g'$ implies the inequality $\|T_f\| \leq \|T_g\|$ (see ([9])). Hence we obtain

Theorem 2.3. *Let g be as Lemma 2.1. If $f \in Co(\alpha)$, then $\|T_F\| \leq \|T_G\|$ where*

$$F(z) = \int_0^z (1 - \zeta)^{\alpha+1} f'(\zeta) d\zeta \text{ and } G(z) = \int_0^z (1 - \zeta)^{\alpha+1} g'(\zeta) d\zeta.$$

Now we state the following corollary:

Corollary 2.4. *For $f \in Co(\alpha)$ and g as in Lemma 2.1, we have*

$$\left| (1 - |z|^2) \frac{f''(z)}{f'(z)} - (\alpha + 1) \frac{1 - |z|^2}{1 - z} \right| \leq \left| (1 - |z|^2) \frac{g''(w(z))}{g'(w(z))} - (\alpha + 1) \frac{1 - |w(z)|^2}{1 - w(z)} \right|,$$

where $w : \mathbb{D} \rightarrow \overline{\mathbb{D}}$ is a holomorphic function with $w(0) = 0$. Equality holds when $w(z) = z$.

Proof. From Lemma 2.1, we have

$$(1 - z)^{\alpha+1} f'(z) \prec (1 - z)^{\alpha+1} g'(z).$$

Using the definition of subordination we have,

$$f'(z) = \frac{(1 - w(z))^{\alpha+1} g'(w(z))}{(1 - z)^{\alpha+1}},$$

where $w : \mathbb{D} \rightarrow \overline{\mathbb{D}}$ is a holomorphic function with $w(0) = 0$. After taking the logarithmic derivative and using the following Schwarz-Pick inequality

$$|w'(z)| \leq \frac{1 - |w(z)|^2}{1 - |z|^2},$$

we get the desired inequality stated in the corollary. Also it is easy to see that equality holds in the inequality when $w(z) = z$. \square

Now, for $f \in Co(\alpha)$, we find the exact set of variability for the functional $(1 - |z|^2)T_f(z)$, which essentially gives both sharp upper and lower bounds for the pre-Schwarzian norm $\|T_f\|$.

Theorem 2.5. *Let $\alpha \in (1, 2]$ be fixed. Then the set of variability of the functional $(1 - |z|^2)T_f(z)$, $f \in Co(\alpha)$, is the closed disk with center*

$$2\bar{z} + (\alpha + 1)(1 - \bar{z})/(1 - z)$$

and radius $\alpha - 1$. The points on the boundary of this disk are attained if and only if f is one of the functions g_θ , where,

$$g_\theta(z) = \frac{1}{\alpha(1 + e^{i\theta})} \left[\left(\frac{1 + e^{i\theta}z}{1 - z} \right)^\alpha - 1 \right], \quad \text{for } \theta \in [0, 2\pi] \setminus \{\pi\},$$

and

$$g_\pi(z) = \frac{z}{1 - z}, \quad \text{for } \theta = \pi.$$

Proof. We use the characterization (1.1) for functions in $Co(\alpha)$ and the representation

$$P_f(z) = \frac{1 - z\omega(z)}{1 + z\omega(z)},$$

where $\omega : \mathbb{D} \rightarrow \overline{\mathbb{D}}$ is an unimodular bounded analytic function. It follows that

$$T_f(z) = \frac{(\alpha - 1)\omega(z) + (\alpha + 1) + 2z\omega(z)}{(1 - z)(1 + z\omega(z))}.$$

By a routine computation one recognizes that

$$(1 - |z|^2)T_f(z) - 2\bar{z} - (\alpha + 1)\frac{1 - \bar{z}}{1 - z} = (\alpha - 1)\frac{\bar{z} + \omega(z)}{1 + z\omega(z)}.$$

Hence, the condition $|\omega(z)| \leq 1$ is equivalent to

$$(2.6) \quad \left| (1 - |z|^2)T_f(z) - 2\bar{z} - (\alpha + 1)\frac{1 - \bar{z}}{1 - z} \right| \leq \alpha - 1.$$

This proves the first part of the assertion in the theorem. The second part follows from the fact that $|\omega(z)| = 1$ if and only if $\omega(z) \equiv e^{i\theta}$, $\theta \in [0, 2\pi]$, and that the solution of the differential equation (1.1) in this case is given by $f(z) = g_\theta(z)$. The

relation between boundary points of the above circle and the extremal function becomes clear from the identity

$$(1 - |z|^2) \frac{g''_{\theta}(z)}{g'_{\theta}(z)} - 2\bar{z} - (\alpha + 1) \frac{1 - \bar{z}}{1 - z} = (\alpha - 1) \frac{e^{i\theta} + \bar{z}}{1 + e^{i\theta}z}.$$

This completes the proof of the theorem. \square

Remark. We remark here that for $f \in Co(\alpha)$, the sharp inequality (2.6) was obtained by Cruz and Pommerenke in [6, Theorem 3]. Their result proves only a one way implication, namely the condition on the disk of variability of the pre-Schwarzian is necessary for f to belong to $Co(\alpha)$. In our theorem we have actually shown that this condition is not only necessary for f to belong to $Co(\alpha)$ but is also sufficient.

Corollary 2.7. *Let $f \in Co(\alpha)$, $\alpha \in [1, 2]$. Then, $4 \leq \|T_f\| \leq 2\alpha + 2$. The equality holds in lower estimate for the function g_{π} and in upper estimate for the function g_0 which are described in the statement of the above theorem.*

Proof. Since

$$\sup_{|z|=r} \left| 2\bar{z} + (\alpha + 1) \frac{1 - \bar{z}}{1 - z} \right| = 2r + 1 + \alpha,$$

where the maximum is attained for $z = r$, we deduce immediately from (2.6), that

$$2 + 2r \leq \sup_{|z| \leq r} (1 - |z|^2) |T_f(z)| \leq 2\alpha + 2r.$$

The lower bound is attained for $f = g_{\pi}$, and the upper bound for $f = g_0$. Indeed, we see that

$$\sup_{|z| \leq r} (1 - |z|^2) |T_{g_{\pi}(z)}| = 2 \sup_{|z| \leq r} \frac{1 - |z|^2}{|1 - z|} = 2(1 + r)$$

and

$$\sup_{|z| \leq r} (1 - |z|^2) |T_{g_0(z)}| = \sup_{|z| \leq r} \frac{|2\alpha + 2z|(1 - |z|^2)}{|1 - z^2|} = 2(\alpha + r).$$

Now, letting $r \rightarrow 1$, we get the sharp estimates

$$4 \leq \|T_f\| \leq 2\alpha + 2, \quad f \in Co(\alpha).$$

\square

Remark. It is well-known that for the class \mathcal{K} of convex univalent functions f , the pre-Schwarzian norm $\|T_f\|$ satisfies the sharp inequality $\|T_f\| \leq 4$ and the equality holds for the convex function $g_{\pi}(z) = z/(1 - z)$. Moreover, we observe that $\|T_f\| \geq 4$ for the class of concave functions and the equality holds for the function $g_{\pi}(z) = z/(1 - z)$ which is common to both the classes and the only function in $Co(\alpha)$ with $\alpha = 1$.

As a consequence of Theorem 2.5, we can obtain a distortion theorem.

Theorem 2.8 (Distortion Theorem). *Let $\alpha \in (1, 2]$. Then, for each $f \in Co(\alpha)$, we have*

$$\frac{(1-r)^{\alpha-1}}{(1+r)^{\alpha+1}} \leq |f'(z)| \leq \frac{(1+r)^{\alpha-1}}{(1-r)^{\alpha+1}}, \quad |z| = r < 1.$$

For each $z \in \mathbb{D}$, $z \neq 0$, equality occurs if and only if $f = g_\theta$, where $\theta \in [0, 2\pi) \setminus \{\pi\}$.

Proof. In view of the inequality (2.6), it follows easily that

$$\left| \frac{zf''(z)}{f'(z)} - \frac{2r^2}{1-r^2} \right| \leq \frac{2\alpha r}{1-r^2}, \quad |z| = r < 1.$$

A standard argument (see for eg. [7, Theorem 2.5]) gives the desired estimate for $|f'(z)|$. Also the sharpness part is easy to verify and so, we skip the routine calculation. \square

In order to include an inclusion result, we need to introduce another notation. Let H^p , $p \in (0, \infty)$, denote the standard Hardy space of analytic functions on the unit disk \mathbb{D} (see for eg. Duren [7, p. 60–62]). It is wellknown that \mathcal{S} is included in H^p for $0 < p < 1/2$. For the class of convex functions, the range for p can be extended to $0 < p < 1$.

Corollary 2.9. *$Co(\alpha) \subset H^p$ for $0 < p < 1/\alpha$. The result is best possible.*

Proof. We fix $z = re^{i\theta}$ with $0 < r < 1$. As $f(0) = 0$, we observe

$$f(z) = \int_0^r f'(\rho e^{i\theta}) e^{i\theta} d\rho.$$

Hence by the distortion theorem and a mild computation, one has

$$|f(z)| \leq \int_0^r \frac{(1+\rho)^{\alpha-1}}{(1-\rho)^{\alpha+1}} d\rho \leq \frac{K}{(1-r)^\alpha},$$

for some positive constant K . The desired result follows from the last inequality and the Prawitz' theorem (see for eg. [7, Theorem 2.22]). \square

There has been a number of investigations on basic subclasses of univalent functions by fixing the second coefficient of functions in these classes. Therefore, it is natural to obtain an analog of Theorem 2.5 for functions in $f \in Co(\alpha)$ with fixed second coefficient. Our next result gives the set of variability of the functional $(1 - |z|^2)T_f(z)$ for $f \in Co(\alpha)$ whenever $f''(0)$ is fixed.

Theorem 2.10. *Let $f \in Co(\alpha)$, $\alpha \in (1, 2]$. Then the set of variability of the functional $T_f(z)(1 - |z|^2)$, $f \in Co(\alpha)$, whenever $f''(0) = \alpha + 1 + (\alpha - 1)a$ with $a \in \overline{\mathbb{D}}$ being fixed, is the disk*

$$\begin{aligned} & \left| (1 - |z|^2)T_f(z) - 2\bar{z} - (\alpha + 1)\frac{1 - \bar{z}}{1 - z} - (\alpha - 1)\frac{\bar{z}(1 + |a|^2 + \bar{a}z) + a}{1 + |z|^2 + 2\operatorname{Re}(az)} \right| \\ & \leq (\alpha - 1)\frac{(1 - |a|^2)|z|}{1 + |z|^2 + 2\operatorname{Re}(az)}. \end{aligned}$$

Proof. As in the proof of Theorem 2.5, a calculation reveals that for $f \in Co(\alpha)$,

$$(2.11) \quad (1 - |z|^2)T_f(z) - 2\bar{z} - (\alpha + 1)\frac{1 - \bar{z}}{1 - z} = (\alpha - 1)\frac{\bar{z} + \omega(z)}{1 + z\omega(z)},$$

where $\omega : \mathbb{D} \rightarrow \overline{\mathbb{D}}$ is an unimodular bounded analytic function.

We see that by (1.1) fixing $f''(0)$ is equivalent to fixing $\omega(0)$, where ω is as above. Indeed we have $f''(0) = \alpha + 1 + (\alpha - 1)\omega(0)$. Now, let

$$(2.12) \quad \omega(z) = \frac{\omega(0) + z\phi(z)}{1 + \overline{\omega(0)}z\phi(z)},$$

where $\phi : \mathbb{D} \rightarrow \overline{\mathbb{D}}$ is again an analytic unimodular bounded function. For convenience, we let $\omega(0) = a$. Then from (2.12), we get

$$z\phi(z) = \frac{\omega(z) - a}{1 - \bar{a}\omega(z)}$$

and a computation shows that $|\phi(z)| \leq 1$ if and only if

$$(2.13) \quad |\omega(z) - W_0| \leq R, \quad z \in \mathbb{D},$$

where

$$W_0 = \frac{a(1 - |z|^2)}{1 - |z|^2|a|^2} \quad \text{and} \quad R = \frac{|z|(1 - |a|^2)}{1 - |z|^2|a|^2}.$$

In order to complete the proof, we let

$$W = \frac{\bar{z} + \omega(z)}{1 + z\omega(z)}.$$

This gives

$$\omega(z) = \frac{W - \bar{z}}{1 - zW}$$

so that (2.13) is equivalent to

$$\left| \frac{W - \bar{z}}{1 - zW} - W_0 \right| \leq R.$$

By a routine calculation the last inequality reduces to

$$(2.14) \quad \left| W - \frac{(1 + \overline{W_0}\bar{z})(\bar{z} + W_0) - R^2\bar{z}^2}{|1 + W_0z|^2 - R^2|z|^2} \right| \leq \frac{R(1 - |z|^2)}{|1 + W_0z|^2 - R^2|z|^2}.$$

An easy exercise gives

$$|1 + W_0z|^2 - R^2|z|^2 = \left(\frac{1 - |z|^2}{1 - |a|^2|z|^2} \right) (1 + |z|^2 + 2\operatorname{Re}(az)),$$

and

$$(1 + \overline{W_0}\bar{z})(\bar{z} + W_0) - R^2\bar{z}^2 = \frac{1 - |z|^2}{1 - |a|^2|z|^2} (\bar{z}(1 + |a|^2 + \bar{a}\bar{z}) + a).$$

Using the above two equalities, we see that the inequality (2.14) takes the following equivalent form

$$\left| W - \frac{\bar{z}(1 + |a|^2 + \bar{a}\bar{z}) + a}{1 + |z|^2 + 2\operatorname{Re}(az)} \right| \leq \frac{(1 - |a|^2)|z|}{1 + |z|^2 + 2\operatorname{Re}(az)}.$$

Hence from (2.11) we get that the set of variability of the functional $T_f(z)(1 - |z|^2)$ is

$$(2.15) \quad \left| (1 - |z|^2)T_f(z) - 2\bar{z} - (\alpha + 1)\frac{1 - \bar{z}}{1 - z} - (\alpha - 1)\frac{\bar{z}(1 + |a|^2 + \bar{a}z) + a}{1 + |z|^2 + 2\operatorname{Re}(az)} \right| \\ \leq (\alpha - 1)\frac{(1 - |a|^2)|z|}{1 + |z|^2 + 2\operatorname{Re}(az)}$$

(where $a = \omega(0)$ is fixed). Whenever $f''(0) = \alpha + 1 + (\alpha - 1)e^{i\theta}$, i.e. $a = e^{i\theta}$, the last inequality reduces to

$$(1 - |z|^2)T_f(z) - 2\bar{z} - (\alpha + 1)\frac{1 - \bar{z}}{1 - z} = (\alpha - 1)e^{i\theta} \left(\frac{e^{i\theta} + \bar{z}}{1 + ze^{i\theta}} \right)^2.$$

As

$$(1 - |z|^2)\frac{g''_{\theta}(z)}{g'_{\theta}(z)} - 2\bar{z} - (\alpha + 1)\frac{1 - \bar{z}}{1 - z} = (\alpha - 1)\frac{e^{i\theta} + \bar{z}}{1 + ze^{i\theta}},$$

the boundary of the disk of variability is attained if and only if $f = g_{\theta}$ where g_{θ} is given in Theorem 2.5. \square

Corollary 2.16. *Let $f \in Co(\alpha)$ and $f''(0) = \alpha + 1$ be fixed. Then,*

$$3 + \alpha \leq \|T_f\| \leq 2 + 2\alpha.$$

Proof. Setting $a = \omega(0) = 0$ in Theorem 2.10 we get

$$\left| (1 - |z|^2)T_f(z) - 2\bar{z} - (\alpha + 1)\frac{1 - \bar{z}}{1 - z} - (\alpha - 1)\frac{\bar{z}}{1 + |z|^2} \right| \leq (\alpha - 1)\frac{|z|}{1 + |z|^2}.$$

This inequality easily gives the required estimates for the pre-Schwarzian norm. \square

3. CONVOLUTION CHARACTERIZATION AND COEFFICIENT ESTIMATES

If $f, g \in \mathcal{H}$, with

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \quad \text{and} \quad g(z) = \sum_{n=0}^{\infty} b_n z^n,$$

then the Hadamard product (or convolution) of f and g is defined by the function

$$(f \star g)(z) = \sum_{n=0}^{\infty} a_n b_n z^n.$$

Clearly, $f \star g \in \mathcal{H}$. In view of the Hadamard convolution, it is now possible to present a new characterization for functions in the class $Co(\alpha)$. The following result will be useful although we did not gain much inroads in this direction.

Theorem 3.1. *Let $1 < \alpha \leq 2$. Then, $f \in Co(\alpha)$ if and only if*

$$(3.2) \quad \frac{1}{z} \left[f \star \frac{(\alpha - 1)z - (\alpha + 1 + 2x)z^2}{(1 - z)^3} \right] + \left[f \star \frac{((\alpha + 1)x + 2)z - (\alpha - 1)xz^2}{(1 - z)^3} \right] \neq 0$$

for all $|z| < 1$ and for all x with $|x| = 1$. Equivalently, this holds if and only if

$$(3.3) \quad \sum_{n \geq 0} A_n z^n \neq 0, \quad A_0 = \alpha - 1, \quad (z \in \mathbb{D}, |x| = 1)$$

where

$$(3.4) \quad f(z) = z + \sum_{n \geq 2} a_n z^n$$

and

$$A_n = (\alpha - n - 1 - nx)(n + 1)a_{n+1} + [n + 1 + (n + \alpha)x]na_n \quad (n \geq 1, a_1 = 1).$$

Proof. We recall $f \in Co(\alpha)$ if and only if $\operatorname{Re} P_f(z) > 0$ in \mathbb{D} , where

$$P_f(z) = \frac{2}{\alpha - 1} \left(\frac{\alpha + 1}{2} \frac{1 + z}{1 - z} - \frac{zg'(z)}{g(z)} \right)$$

with $g(z) = zf'(z)$. We note that P_f is analytic in \mathbb{D} with $P_f(0) = 1$. Thus, $f \in Co(\alpha)$ is equivalent to

$$P_f(z) \neq \frac{x - 1}{x + 1} \quad (z \in \mathbb{D}, |x| = 1, x \neq -1)$$

which, by a simplification, is same as writing

$$(3.5) \quad \left(\frac{x + \alpha}{x + 1} \right) \frac{g(z)}{z} + \left(\frac{\alpha x + 1}{x + 1} \right) g(z) - g'(z) + zg'(z) \neq 0.$$

Recall that

$$\frac{g(z)}{z} = \frac{g(z)}{z} \star \frac{1}{1 - z}, \quad zg'(z) = g(z) \star \frac{z}{(1 - z)^2}, \quad f'(z) \star p(z) = \frac{f(z)}{z} \star (zp)'(z).$$

Using these identities, $g(z) = zf'(z)$, (3.5) gives that $f \in Co(\alpha)$ if and only if

$$f'(z) \star \left(\frac{(x + \alpha)(1 - z) - (x + 1)}{(1 - z)^2} \right) + z \left[f'(z) \star \left(\frac{(\alpha x + 1)(1 - z) + (x + 1)}{(1 - z)^2} \right) \right] \neq 0.$$

After some simplification the above takes the following equivalent form

$$(3.6) \quad \begin{aligned} & \frac{f(z)}{z} \star \left[\frac{\alpha - 1 - (\alpha + 1 + 2x)z}{(1 - z)^3} \right] \\ & + z \left[\frac{f(z)}{z} \star \left(\frac{(\alpha + 1)x + 2 - (\alpha - 1)xz}{(1 - z)^3} \right) \right] \neq 0 \end{aligned}$$

which gives (3.2). To obtain the series formulation of it, we first observe that

$$\begin{aligned} p(z) &= \frac{\alpha - 1 - (\alpha + 1 + 2x)z}{(1 - z)^3} \\ &= \alpha - 1 + \sum_{n \geq 1} (n + 1) [\alpha - n - 1 - nx] z^n \end{aligned}$$

and

$$\begin{aligned} q(z) &= \frac{((\alpha + 1)x + 2)z - (\alpha - 1)xz^2}{(1 - z)^3} \\ &= ((\alpha + 1)x + 2)z + \sum_{n \geq 2} n[n + 1 + (n + \alpha)x]z^n. \end{aligned}$$

Using these two identities, (3.6) can be written in terms of convolution as follows: $f \in Co(\alpha)$ ($1 < \alpha \leq 2$) if and only if

$$\frac{f(z)}{z} \star p(z) + f(z) \star q(z) \neq 0$$

which is same as (3.3). We complete the proof. \square

In order to reveal the interaction between the class $Co(\alpha)$ and wellknown Kaplan class, we need to introduce the following definition.

Definition 3.1. A nonvanishing analytic function s in \mathbb{D} with $s(0) = 1$ is said to belong to the *Kaplan class* $K(\alpha, \beta)$ ($\alpha \geq 0, \beta \geq 0$) if for $0 < r < 1$ and $\theta_1 < \theta_2 < \theta_1 + 2\pi$ we have

$$-\alpha\pi \leq \int_{\theta_1}^{\theta_2} \left\{ \operatorname{Re} \frac{re^{i\theta} s'(re^{i\theta})}{s(re^{i\theta})} - \frac{1}{2}(\alpha - \beta) \right\} d\theta \leq \beta\pi.$$

Following the notation of Sheil-Small [14], for λ real we consider

$$\Pi_\lambda = \begin{cases} K(\lambda, 0) & (\lambda \geq 0) \\ K(0, -\lambda) & (\lambda < 0). \end{cases}$$

This gives $f \in \Pi_\lambda$ if and only if for $z \in \mathbb{D}$,

$$\operatorname{Re} z \frac{s'(z)}{s(z)} \begin{cases} < \frac{1}{2}\lambda & (\lambda > 0) \\ > \frac{1}{2}\lambda & (\lambda < 0). \end{cases}$$

The class $\Pi_0 = K(0, 0)$ contains the constant function $s(z) = 1$ only (compare [14]).

Theorem 3.7. Let $\alpha \in (1, 2]$. A function $f \in Co(\alpha)$ if and only if there exists a function $s \in \Pi_{\alpha-1}$ such that

$$(3.8) \quad f(z) = \int_0^z \frac{s(t)}{(1-t)^{\alpha+1}} dt.$$

Proof. The function f given by (3.8) satisfies

$$s(z) = (1-z)^{\alpha+1} f'(z).$$

A computation from (1.1) shows that

$$(3.9) \quad \frac{s'(z)}{s(z)} = \frac{f''(z)}{f'(z)} - \frac{\alpha+1}{1-z} = \frac{\alpha-1}{2} \frac{1-P_f(z)}{z}.$$

This gives

$$\operatorname{Re} \left(z \frac{s'(z)}{s(z)} \right) < \frac{\alpha-1}{2}, \quad z \in \mathbb{D}$$

if and only if

$$\operatorname{Re} \left[\frac{(\alpha + 1)}{2} \frac{1 + z}{1 - z} - 1 - z \frac{f''(z)}{f'(z)} \right] > 0, \quad z \in \mathbb{D}.$$

The proof follows. \square

The functions s defined as above produces a simple characterization in terms of the Hadamard product.

Theorem 3.10. *A function $f \in Co(\alpha)$ if and only if*

$$s(z) \star \left(\frac{z}{(1 - z)^2} + \frac{1 - \alpha}{x + 1} \frac{1}{1 - z} \right) \neq 0, \quad z \in \mathbb{D}, \quad |x| = 1, x \neq -1,$$

for some $s \in \Pi_{\alpha-1}$.

Proof. We recall that, $f \in Co(\alpha)$ is equivalent to

$$P_f(z) \neq \frac{x - 1}{x + 1} \quad (z \in \mathbb{D}, \quad |x| = 1, \quad x \neq -1),$$

which by (3.9) is equivalent to

$$\frac{-2}{\alpha - 1} \left(\frac{zs'(z)}{s(z)} \right) + 1 \neq \frac{x - 1}{x + 1}.$$

A simplification gives

$$(1 - \alpha) \left(s(z) \star \frac{1}{1 - z} \right) + (x + 1) \left(s(z) \star \frac{z}{(1 - z)^2} \right) \neq 0$$

and the desired condition follows. \square

Moreover, if $f \in Co(\alpha)$, we may define a function $\varphi : \mathbb{D} \rightarrow \overline{\mathbb{D}}$ by

$$(3.11) \quad P_f(z) = \frac{1 + z\varphi(z)}{1 - z\varphi(z)}.$$

In view of this and (3.9), we have

$$(3.12) \quad s'(z) = \varphi(z)((1 - \alpha)s(z) + zs'(z)).$$

We want to find bounds for the moduli of the Taylor coefficients b_k , $k \in \mathbb{N}$, of the function s that are defined via the series representation

$$s(z) = \sum_{k=0}^{\infty} b_k z^k, \quad b_0 = 1.$$

To that end we use Theorem 2.2 in [11] (compare also [5, 12, 13]). Using these methods, we see that (3.12) implies the inequalities

$$\sum_{k=1}^N k^2 |b_k|^2 \leq \sum_{k=0}^{N-1} (k + 1 - \alpha)^2 |b_k|^2, \quad N \in \mathbb{N}.$$

Since $k + 1 - \alpha < k$, we may use mathematical induction to prove the inequality

$$(3.13) \quad |b_k| \leq \frac{\alpha - 1}{k}, \quad k \in \mathbb{N}.$$

Equality in (3.13) can be achieved if and only if

$$b_m = 0, \quad m = 1, \dots, k - 1.$$

If we insert this into (3.12) and assume equality in (3.13), we recognize that this implies

$$\varphi(z) = e^{i\theta} z^{k-1}, \quad \theta \in [0, 2\pi].$$

Solving (1.1) with P_f defined by (3.11) and this function φ leaves us with the fact that the functions

$$(3.14) \quad f'(z) = \frac{(1 - e^{i\theta} z^k)^{\frac{\alpha-1}{k}}}{(1 - z)^{\alpha+1}}$$

are the unique extremal functions for the inequalities (3.13). The Schwarz-Christoffel formula implies that the functions (3.14) deliver functions in $Co(\alpha)$ that map the unit disk conformally onto the complement of an unbounded polygon with k or $k - 1$ finite vertices.

Remark. Since we may consider the functional that maps f into $s^{(k)}(0)/k!$ as a linear functional on $Co(\alpha)$ with the set of variability described by (3.13) with a unique extremal function corresponding to any boundary point, we get new examples supporting a conjecture formulated in [4]. There, we conjectured that any conformal map of the unit disk onto the complement of an unbounded convex polygon is an extremal point of the closed convex hull of $Co(\alpha)$.

In view of the discussion above one can quickly get the following

Theorem 3.15. *Let $f \in Co(\alpha)$ have the expansion (3.4). Then the following sharp inequality holds*

$$(3.16) \quad \left| \sum_{k=0}^n (-1)^k \binom{\alpha+1}{n-k} (k+1) a_{k+1} \right| \leq \frac{\alpha-1}{n} \quad (n \geq 1).$$

In particular, we have

$$\begin{aligned} \text{(i)} \quad & \left| a_2 - \frac{\alpha+1}{2} \right| \leq \frac{\alpha-1}{2} \\ \text{(ii)} \quad & \left| 3a_3 - 2(\alpha+1)a_2 + \frac{\alpha(\alpha+1)}{2} \right| \leq \frac{\alpha-1}{2}. \end{aligned}$$

Proof. We deduce from Theorem 3.7 that $f \in Co(\alpha)$ if and only if $s(z) \in \Pi_{\alpha-1}$ where

$$s(z) = (1 - z)^{\alpha+1} f'(z).$$

Comparing the coefficient z^n in the series expansion of the functions involved above, we obtain that

$$b_n = \sum_{k=0}^n (-1)^k \binom{\alpha+1}{n-k} (k+1) a_{k+1}$$

and the desired inequality (3.16) follows, if we use the estimate (3.13). The two particular cases follow, if we let $n = 1, 2$. Also the estimate is sharp for each n for the functions $f \in Co(\alpha)$ such that

$$f'(z) = \frac{(1 - e^{i\theta} z^n)^{\frac{\alpha-1}{n}}}{(1 - z)^{\alpha+1}}.$$

□

Remark. Case (i) of Theorem 3.15 is wellknown whereas Case (ii) of Theorem 3.15 gives that for $f \in Co(2)$ one has

$$|1 - 2a_2 + a_3| \leq \frac{1}{6}.$$

This result is obtained recently in [15, Theorem 3].

The classical Alexander transform $\int_0^z (f(t)/t) dt$ provides a one-to-one correspondence between \mathcal{S}^* and \mathcal{K} . It is then natural to ask whether a similar correspondence can be established for \mathcal{S}^* and $Co(\alpha)$. The answer is provided by the following result which may be used to study the geometric properties of $\Lambda_\phi(z)$ when ϕ belongs to various subclasses of \mathcal{S} .

Theorem 3.17. *Let $\alpha \in (1, 2]$. A function $f \in Co(\alpha)$ if and only if there exists a $\phi \in \mathcal{S}^*$ such that $f(z) = \Lambda_\phi(z)$, where*

$$\Lambda_\phi(z) = \int_0^z \frac{1}{(1-t)^{\alpha+1}} \left(\frac{t}{\phi(t)} \right)^{(\alpha-1)/2} dt.$$

Proof. In view of Theorem 3.7, it suffices to show that $s \in \Pi_{\alpha-1}$ if and only if there exists a $\phi \in \mathcal{S}^*$ such that

$$\phi(z) = z (s(z))^{2/(1-\alpha)}, \quad z \in \mathbb{D}.$$

However, this fact is clear because

$$\frac{2}{\alpha-1} \operatorname{Re} \left(z \frac{s'(z)}{s(z)} \right) = 1 - \operatorname{Re} \left(z \frac{\phi'(z)}{\phi(z)} \right), \quad z \in \mathbb{D}.$$

□

For $f \in Co(\alpha)$, the above characterization shows that there exists $\phi \in \mathcal{S}^*$ such that

$$(3.18) \quad f'(z) = \frac{1}{(1-z)^{\alpha+1}} \left(\frac{z}{\phi(z)} \right)^{(\alpha-1)/2}.$$

Set

$$f(z) = z + \sum_{n \geq 2} a_n z^n \quad \text{and} \quad \phi(z) = z + \sum_{n \geq 2} \phi_n z^n.$$

A comparison of coefficients of z^2 on both side of (3.18) yields

$$3a_3 = -\frac{\alpha-1}{2}\phi_3 + \frac{(\alpha+1)(\alpha-1)}{8}\phi_2^2 - \frac{\alpha^2-1}{2}\phi_2 + \frac{(\alpha+1)(\alpha+2)}{2}.$$

That is,

$$\frac{3}{\alpha^2-1} \left(a_3 - \frac{(\alpha+1)(\alpha+2)}{6} \right) = A(\phi_3, \phi_2, \alpha)$$

where

$$A(\phi_3, \phi_2, \alpha) = -\frac{1}{2(\alpha+1)} \left[\phi_3 - \frac{\alpha+1}{4}\phi_2^2 + (\alpha+1)\phi_2 \right].$$

In view of the result of Avkhadiev and Wirths [2, Corollary 3], one has the following result which does not seem to be known in this form.

Corollary 3.19. *Let $\phi(z) = z + \sum_{n \geq 2} \phi_n z^n$ belong to \mathcal{S}^* and $\alpha \in (1, 2]$. Then $A(\phi_3, \phi_2, \alpha) \in \overline{h(\mathbb{D})}$, where*

$$h(z) = z + \frac{\alpha-2}{2(\alpha+1)}z^2.$$

It is not clear whether the present restriction on α is essential in the last corollary.

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